

Chapter 8

The Economic Order-Quantity (EOQ) Model

Leroy B. Schwarz
Purdue University

The economic order-quantity model considers the tradeoff between ordering cost and storage cost in choosing the quantity to use in replenishing item inventories. A larger order-quantity reduces ordering frequency, and, hence ordering cost/month, but requires holding a larger average inventory, which increases storage (holding) cost/month. On the other hand, a smaller order-quantity reduces average inventory but requires more frequent ordering and higher ordering cost/month. The cost-minimizing order-quantity is called the Economic Order Quantity (EOQ). This chapter builds intuition about the robustness of EOQ, which makes the model useful for management decision-making even if its inputs (parameters) are only known to be within a range of possible values. This chapter also provides intuition about choosing an inventory-management system, not just an EOQ.

Introduction

Lauren Worth is excited about her new job at Cardinal Hospital. Lauren had worked at Cardinal, first as a candy-striper, and then, after college, as a registered nurse. After several years “working the wards,” Lauren left Cardinal to get an MBA. Now, having recently graduated, Lauren had returned to Cardinal as its first “Inventory Manager.”

Lauren’s boss, Lee Atwood, is Purchasing Manager at Cardinal Hospital. Lauren has known Lee since her days as a candy-striper, and regards him as a friend. Lee describes himself as being from the “old school” of inventory management. “Lauren, I know that there are sophisticated methods for managing inventories, and, in particular for determining ‘optimal order-quantities.’ And, I know that Cardinal is paying a ‘management-cost penalty’ for using other than optimal order-quantities. In other words, that Cardinal is incurring higher than the minimum possible inventory-management costs in managing its inventories. But, frankly, I’m skeptical about what it will cost Cardinal Hospital to reduce that management-cost penalty. That’s because determining the optimal order-quantity will require two things that are in short supply around here.”

“First, is the knowledge about costs and demands. For example, in the general supplies area, where I would like you to begin, in order to use the ‘EOQ model,’ you need to know what it costs to place an order, and how much it costs to hold an item in inventory. Now, I know that it costs Cardinal money to place an order and money to hold an item in inventory, but we don’t know what these costs are. Given enough time and trouble, we can find out what those numbers are, but, at best, they would be estimates. Now, let’s suppose it costs, say, \$100 a year to get those estimates for each of the 2,000 general-supply items we are talking about, and to keep those numbers up to date. Once we plug those numbers into this formula and determine the ‘optimal’ order-quantity, will using that order-quantity save us at least the \$100/year that it is costing us to feed that formula? In other words, if we save \$90/year by making better decisions, but it costs us \$100/year to make those better decisions, then we have lost \$10/year!”

“Another thing that’s been in short supply, at least until you’ve arrived, is enough expertise to know which inventory-management system we should use to manage the different inventories we manage. For example, as you know, some of these items are expensive, others are very inexpensive; some perish, some don’t; some become obsolete quickly, others don’t. I know there are lots of different systems out there—EOQ, JIT, MRP, and the ‘newsvendor,’ to name just a few—but each of these has overhead connected with it. It is logical that the more sophisticated the model, the ‘better’ the order-quantity decision, but it also makes common sense that the more sophisticated model will be more expensive to install and keep up to date. For example, suppose that the EOQ model would reduce ordering and holding costs by \$200/year, but cost us only \$100/year to use and keep up to date: a net saving of \$100/year. Would some other model—JIT, for example—net Cardinal a larger saving?”

Lauren was confused. She had thought that all she had to do was to plug numbers into the EOQ model and Cardinal would be making optimal order-quantity decisions. She had not thought about the cost of getting those numbers; or, worse, the fact that those numbers would only be estimates that would have to be updated over time. As a consequence, it had not occurred to her that Lee Atwood’s policy of ordering a 3-month supply for *everything* might, all things considered, be a good decision-rule. Finally, Lauren also had not considered the possibility of using a more sophisticated system than EOQ, one that might yield larger savings but have a higher overhead cost.

The Decisions to be Made

One of the most frequent decisions faced by operations managers is “how much” or “how many” of something to make or buy in order to satisfy external (e.g., customer demand) or internal requirements for some item.

Many times, this decision is made with little or no thought about its cost consequences. Examples are “order a case” or “make a month’s supply.” Lee Atwood’s

policy is to order a 3-month supply whenever he ordered anything. In *some* business scenarios, such decision-rules might be OK, in the sense that they require very little information (e.g., what *is* a month's supply?), are easy to implement, and yield inventory-management costs that are only a few percent above the minimum possible cost (i.e., a small management-cost penalty). Indeed, such “thoughtless” decisions might even be optimal, in the sense that they might minimize the *total* cost of inventory management, including the cost of the information and implementation requirements of the decision-rule being used. However, in other scenarios, decision-rules like these can be quite bad; in other words, their associated management cost-penalty is very large and, an alternative decision-rule, one that required more expensive information and/or implementation would yield a considerably smaller total cost of inventory management.

So, the *fundamental* decision about the “order-quantity decision” *isn't* “What's the right quantity”? Instead, it is: “Under what circumstances does the *choice* of the order-quantity make a lot of difference (in terms of the management-cost penalty)?” After all, if the choice of the order-quantity does not make a significant difference, then common sense suggests that the choice of the order-quantity does not matter much; and, therefore, should be made with a focus on making and implementing the order-quantity decision as inexpensively as possible (in terms of its information and implementation costs). For example, at Cardinal Hospital, Lee Atwood's 3-month supply policy might be fine; that is, using EOQ—or some even more sophisticated system—would cost more than it was worth. On the other hand, if the choice of the order-quantity *does* make a significant difference, then the order-quantity decision may deserve a great deal of management attention and thoughtful judgment. Or, to quote Ford Harris (1913), who created the EOQ model, “This is a matter that calls, in each case, for a trained judgment, for which there is no substitute.”

This chapter will address the question of whether or not the choice of the order-quantity makes a lot of difference in terms of its management-cost penalty, using the “Economic Order Quantity (EOQ) model.”

Assuming that the order-quantity decision *does* make a significant difference, the question *then* becomes “What's the right quantity”? As might be expected, the larger the management-cost penalty, the more difficult it is to answer this question, typically because of costs and/or constraints that arise from the complexity of each specific business scenario. In such scenarios, the insights provided by the analysis of the EOQ model (see below) and the so-called “EOQ formula” are often helpful in guiding management about the order-quantity decision. Or, to quote Ford Harris again, in such scenarios, “... using the formula as a check, is at least warranted.”

The EOQ Business Scenario

The EOQ business scenario is simple. It involves selecting the order-quantity “Q” that minimizes the average inventory-management cost/time—where “time” can be a “year,” a “month,” or a “week”— for an item with demand that goes on forever

at a demand *rate* that never changes. Pretty simple, right? So simple, that if the value of the insights it provides depended on being able to find even *one* real-world scenario that matched it, then it would be worthless. But, therein lies the tale....

Customer Demand

The customer demand (or internal usage, as at Cardinal Hospital) rate of the item is assumed to be known and constant at a rate D units/time. The “units” dimension might be cases or gallons or truckloads; it does not matter. For convenience we will measure D in units of one and time in years.

Leadtime

The EOQ business scenario also assumes that the order (or manufacturing) leadtime (i.e., the time interval between placing the order and receiving the corresponding order quantity) is zero. In other words, that delivery or manufacturing is instantaneous. This assumption (conveniently) removes the question of “When to order?” the chosen order-quantity: Order Q each time inventory falls to zero.

Although the instantaneous-delivery assumption is obviously unrealistic—after all, even in Star Trek movies the Replicator takes a few moments to produce what Captain Picard orders—any *fixed* leadtime is easily accommodated, provided that demand is known and constant: Order Q when inventory equals a leadtime’s supply. For example, if the leadtime is one week, then order Q when inventory equals a week’s supply.

On the other hand, if leadtimes are unpredictable, then even given a known and constant demand rate, D , the question “When to order?” becomes much more difficult to answer. We will address this question briefly below.

Costs

The EOQ business scenario accounts for three types of costs: (1) the cost of the units themselves; (2) the cost of holding units in inventory; and (3) the fixed order (or manufacturing set-up) cost.

Unit Cost: The cost of the units themselves, denoted c , and measured in \$/unit, is assumed to be fixed, regardless of the number of units ordered or manufactured; although quantity discounts or other economies of scale are ignored in the *basic* EOQ model, but are easily accommodated (See below).

Inventory-Holding Cost: The inventory-holding cost in the EOQ business scenario, denoted h , represents management’s cost of capital (i.e., the time value of

money) invested in the units, the cost of the space consumed by the units, taxes or insurance premiums on the inventory itself, and, in some cases, allowances for obsolescence (i.e., the possibility that the items might become useless while they are sitting in inventory) or “shrinkage” (i.e., that units may be lost or stolen).

The inventory-holding cost, h , is measured in $\$/(\text{unit} \times \text{time})$, where the “unit” and the “time” dimensions are the same as used in defining the demand rate D . So, for example, if the demand rate is measured in gallons/year, then h is measured in $\$/(\text{gallon} \times \text{year})$; in other words, the dollar cost to hold one gallon for one year’s time. For convenience, as above, we will measure time in years.

Every thoughtful inventory manager recognizes that holding inventory costs money. In many scenarios, the value of h is likely to be between 25% and 50% of the cost of the item itself, depending on the company’s cost of capital—for example, what it costs the company to borrow money—and the item’s risk of obsolescence. On the other hand, it is typically very difficult to determine h ’s exact dollar value. For example, does it cost, say, \$1 to hold a unit of an item in inventory for a year; or does it cost \$2? Most company’s cost-accounting systems don’t report this number, although every accountant knows that it’s real. It’s even possible that, over time, the value of h changes. For example, if storage space is ample at some times of the year and tight at others, or if a company’s cost of capital changes over time, then the value of h changes over time.

For the time being, we will assume that the value of h is known and constant, and examine the importance of this assumption below.

Fixed Order Cost: The fixed order cost in the EOQ business scenario, denoted K , and measured in dollars, represents all the costs associated with placing an order *excluding* the cost of the units themselves. In other words, K , represents any administrative (i.e., paperwork) costs of placing and/or receiving an order (e.g., inspection); and, in a manufacturing scenario, the cost of setting up the equipment to produce the order-quantity.

Like inventory-holding costs, fixed order costs are real costs for every company; and, in some scenarios, very significant costs. A set-up (i.e., equipment changeover) in pharmaceutical manufacturing, for example can shut down production and engage dozens of workers for weeks at a time; thereby costing thousands of dollars. However, like inventory-holding cost, the challenge is determining its exact dollar value. For example, does it cost, say, \$100 or \$300 to place an order? Most cost-accounting systems do not report this number, either. It is even possible that, like the inventory-holding cost h , the value of K changes over time. For example, if the ordering, set-up, and/or receiving departments are busy at some times of the year but idle at other times, then the value of K itself changes over time.

For the time being, we will assume that the value of K is known and constant, and examine the importance of this assumption below.

Picking the “Optimal” Order-Quantity: In the EOQ business scenario, the goal is to select the order-quantity, Q , that minimizes the average inventory-management cost/time—measured in $\$/\text{year}$ —that results from ordering Q (whenever inventory reaches zero), and doing that (repetitively) forever (i.e., over an “infinite time horizon”). We will denote this *average cost* resulting from Q as “ $AC(Q)$ ”

Given the three costs accounted for in the EOQ scenario it follows that $AC(Q)$ must be:

$$AC(Q) = K \times \frac{D}{Q} + h \times \frac{Q}{2} + c \times D. \quad (1)$$

That is, as described above, the average cost/year of using Q as the order-quantity has three parts, the average annual cost of ordering (first term), the average annual inventory-holding cost (second term), and the average annual cost of the units themselves.

Taking a closer look, the average annual ordering cost, $K \times (D/Q)$, equals the fixed order cost K , multiplied by the average number of orders placed per year, D/Q . (For example, if the demand rate D is 1,200 units/year and $Q = 200$ units, then, on average $D/Q = 1,200/200 = 6$ orders will be placed each year.) The middle term in (1), $h \times (Q/2)$, is the inventory-holding cost rate, h , multiplied by the average inventory, $Q/2$. (The average inventory is $Q/2$ since the maximum is Q , the minimum is zero, and it falls from Q to zero at a uniform rate.) Finally, the average annual cost of the units themselves is $c \times D$; that is, the cost/unit of the item being inventoried, multiplied by its annual demand.

The first thing to notice about $AC(Q)$ is that the average annual cost of the units themselves *does not depend* on Q ; that is, whether we buy in small quantities (e.g., place several orders per year) or in large quantities (e.g., order every other year), the average cost of the units themselves doesn't depend on the order-quantity Q . Consequently, this term, $c \times D$, is typically left out of $AC(Q)$, as in (2) below:

$$AC(Q) = K \times \frac{D}{Q} + h \times \frac{Q}{2}. \quad (2)$$

Written in this manner, $AC(Q)$ is the sum of two terms, one of them: $K \times (D/Q)$, which decreases as Q increases; and the other, $h \times (Q/2)$, which increases as Q increases. In managerial terms, this makes perfect sense: The average annual ordering cost increases as Q decreases because more orders must be placed each year; and the average annual inventory-holding cost increases as Q increases because the average inventory, $Q/2$, increases. [Figure 8.1](#) provides a graph of $AC(Q)$ and its two components: $K \times (D/Q)$ and $h \times (Q/2)$.

Note that the average ordering cost, $K \times (D/Q)$, decreases as Q increases. For example, if $D = 1,200$ units/year, then as Q increases from, say, 100–1,200, then the average number of orders/year, D/Q , decreases from 12 to 1 time/year. (Notice also, that if, say, $Q = 2,400$, then one order will be placed every 2 years, and $D/Q = 1/2$.) So, if management wanted to minimize the average order cost, then it would choose the largest possible value of Q . For example, this might be the largest Q that it could afford to buy or the most it could store in the space available. On the other hand, the average inventory-holding cost, $h \times (Q/2)$, increases as Q increases. In other words, the larger the order-quantity, the larger the average inventory, and,

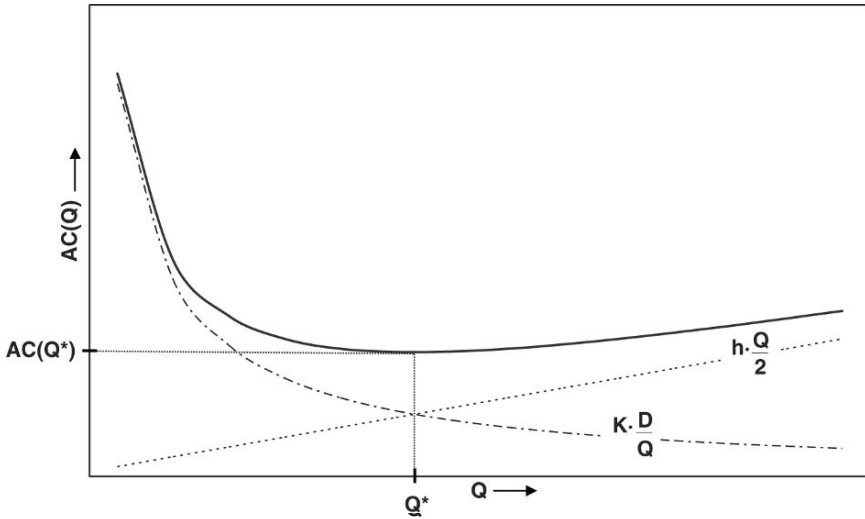


Fig. 8.1 A Graph of $AC(Q)$ Versus Q

hence, the larger the average inventory-holding cost. So, if management wanted to minimize the average inventory-holding cost, then it would order the smallest possible value of Q . For example, this might be the minimum that the supplier would sell or the smallest quantity that could be made.

Hence, the choice of Q is a trade-off between the average ordering cost and the average inventory-holding cost. Note in Fig. 8.1 that the minimum value of $AC(Q)$ occurs where the average annual ordering costs and average annual inventory-holding graphs cross; that is, where $K \times D/Q = h \times (Q/2)$. Thus, we can find the optimal Q , denoted Q^* by solving equation (3) for Q^* ; that is, solving

$$K \times \frac{D}{Q^*} = h \times \frac{Q^*}{2} \tag{3}$$

for Q^* . The solution is:

$$Q^* = \sqrt{\frac{2 \times K \times D}{h}}. \tag{4}$$

This is the so-called “square-root” or EOQ formula for Q^* .

To determine $AC(Q^*)$, the inventory-management cost/year using $Q = Q^*$, one substitutes (4) into (2). After some algebra:

$$AC(Q^*) = \sqrt{2 \times K \times D \times h}. \tag{5}$$

A Closer Look at the Assumptions Of EOQ: Sensitivity Analysis

In order to develop the EOQ model, several unrealistic assumptions must be made. The most *unrealistic* of these is that the demand rate is known and constant. However, even if management were comfortable with that assumption, there is the practical problem of estimating the values of the fixed order cost (K) and the inventory-holding cost (h) to “plug” into [equation \(4\)](#).

In order to assess the impact of all these assumptions, we are going to do “sensitivity analysis.” That is, we are going to assess the consequences of these assumptions on the costs we experience because we are making assumptions that might not be—indeed, probably are not—valid. We are going to conduct this sensitivity analysis in several steps. In Step 1 we will assume that all of the assumptions of the EOQ business scenario are valid, but that we ignore the EOQ formula, and, instead, select Q without much thought. In Steps 2 and 3, we will continue to assume that EOQ model’s assumptions hold, and that we use the EOQ formula, but that we only have very rough estimates about its inputs: K , D , and h . In Step 4 we will use the insight provided in Steps 2 and 3 to examine the model’s assumptions about known and constant demand and known and constant cost parameters. We will continue to assume that production and/or delivery are instantaneous, in order to focus on the order-quantity decision.

Sensitivity Analysis: Step 1

We begin with a business scenario in which *all* of the assumptions of the EOQ business scenario are valid. We even know the true values of D , h , and K ; *but* we select a value of Q without much thought (e.g., order a case). We will denote our choice of Q as \hat{Q} . It is unlikely, of course, that our chosen \hat{Q} will equal Q^* . Since there is only one Q^* that minimizes $AC(Q)$, we will be paying a management-cost penalty for using \hat{Q} instead of Q^* ; that is $AC(\hat{Q}) \geq AC(Q^*)$, and the difference, $[AC(\hat{Q}) - AC(Q^*)]$, is the penalty. For example, if our \hat{Q} is larger than Q^* , then, in [Fig. 8.1](#), \hat{Q} will be to the right of Q^* and we will experience $AC(\hat{Q})$ larger than $AC(Q^*)$. How much higher, of course, depends on how much larger \hat{Q} is than Q^* and on the shape of the $AC(Q)$ function. Suppose, for example that our chosen \hat{Q} is twice the size of Q^* ; i.e., $\hat{Q} = 2 \times Q^*$. The question we want to answer is: *Will the management-cost we pay, $AC(\hat{Q})$, be 2 times $AC(Q^*)$? Will it be more, or will it be less?*

Specifically, we are interested in the ratio $AC(\hat{Q})/AC(Q^*)$. Notice that if, by chance, $\hat{Q} = Q^*$, then, of course, $AC(\hat{Q}) = AC(Q^*)$, and the “penalty-cost ratio” would equal 1. Otherwise, this ratio would be above 1, say, 1.25, which indicates that the management penalty cost is 25% above the minimum possible $AC(Q^*)$. Using [equation \(2\)](#), once with $Q = \hat{Q}$ and once with $Q = Q^*$ we get:

$$\frac{AC(\hat{Q})}{AC(Q^*)} = \frac{K \times \frac{D}{\hat{Q}} + h \times \frac{\hat{Q}}{2}}{K \times \frac{D}{Q^*} + h \times \frac{Q^*}{2}} \tag{6}$$

Using the fact that $K \times (D / Q^*) = h \times (Q^* / 2) = \frac{\sqrt{2 \times K \times D \times h}}{2}$, after a little algebra,

$$\frac{AC(\hat{Q})}{AC(Q^*)} = 0.5 \times \left[\frac{Q^*}{\hat{Q}} + \frac{\hat{Q}}{Q^*} \right] \tag{7}$$

What [equation \(7\)](#) says is that the ratio of the management cost we *will* experience—from using the “wrong” Q, \hat{Q} —to the minimum possible management cost we *would* experience—if we used the optimal Q, Q^* —is one-half the sum of Q^*/\hat{Q} and \hat{Q}/Q^* . So, for example, if our chosen $\hat{Q} = 2 \times Q^*$ then

$$\frac{AC(\hat{Q})}{AC(Q^*)} = 0.5 \times [0.5 + 2] = 1.25.$$

In other words, the average annual cost we will experience will be 25% higher than the minimum possible. Note that a 100% error in choosing \hat{Q} has yielded a management-cost penalty of only 25% of $AC(Q^*)$.

Table 8.1 provides the values of $AC(\hat{Q})/AC(Q^*)$ for \hat{Q} ranging from 0.1 Q^* to 10 Q^* . Note, that the penalty-cost ratio is symmetric around 1 (e.g., that the ratio is the same for $\hat{Q}/Q^* = 0.1$ as for $\hat{Q}/Q^* = 10$).

The management intuition from Step 1 is that as long as \hat{Q} is in the “ballpark” of Q^* (e.g., within a factor of 2), then the penalty-cost ratio is relatively small (e.g., less than or equal to 1.25), whereas if \hat{Q} is “way off” (e.g., more than 10 times larger or smaller), then the penalty-cost ratio will be quite large (e.g., greater than or equal to 5.050). Although this is a *valuable intuitive insight*, its *practical value* is limited by the assumption that we know what Q^* is. This leads to Step 2:

Sensitivity Analysis: Step 2

Consider a closely related business scenario in which: (a) all of the assumptions of the EOQ business scenario are valid; (b) management has chosen to use the EOQ formula to compute the order quantity; (c) management *knows* the exact dollar values of K , the fixed order cost (e.g., $K = \$100$), and h , the inventory-holding cost (e.g., $h = \$2/\text{unit} \times \text{year}$); but (d) *doesn't know* what the demand rate, D , is. Instead, management only knows the range of possible values for D ; for example, that $900 \leq D \leq 1,600$; but doesn't know for certain what the value of D is.

Table 8.1 $AC(\hat{Q})/AC(Q^*)$ Versus \hat{Q}/Q^*

\hat{Q}/Q^*	$AC(\hat{Q})/AC(Q^*)$
0.1	5.050
0.15	3.408
0.2	2.600
0.25	2.125
0.3	1.817
0.35	1.604
0.4	1.450
0.45	1.336
0.5	1.250
0.55	1.184
0.6	1.133
0.65	1.094
0.7	1.064
0.75	1.042
0.8	1.025
0.85	1.013
0.9	1.006
0.95	1.001
1	1.000
1.05	1.001
1.1	1.005
1.25	1.025
1.5	1.083
2	1.250
5	2.600
6	3.083
7	3.571
8	4.063
9	4.556
10	5.050

Let's begin by assuming that the value of $D = 900$ units/year. Using [equation \(4\)](#) yields

$$Q^* = \sqrt{\frac{2 \times K \times D}{h}} = \sqrt{\frac{2 \times 100 \times 900}{2}} = 300 \text{ units/order.}$$

Similarly, if we assume that the value of $D = 1,600$ units/year, then

$$Q^* = \sqrt{\frac{2 \times K \times D}{h}} = \sqrt{\frac{2 \times 100 \times 1600}{2}} = 400 \text{ units/order.}$$

Based on these calculations we *have not learned* the value of Q^* , since we remain ignorant about the value of D . However, we *have learned* that Q^* must be between 300 and 400 units/order!

As a consequence, we know that if we chose $\hat{Q} = 300$ (assuming $D = 900$) when, in fact, $D = 1,600$, and, hence, $Q^* = 400$, then $\hat{Q}/Q^* = 300/400 = 0.75$; and, using [\(7\)](#), the cost-penalty ratio would be:

$$\frac{AC(\check{Q})}{AC(Q^*)} = 0.5 \left[\frac{Q^*}{\check{Q}} + \frac{\check{Q}}{Q^*} \right] = 0.5 \times \left[0.75 + \frac{1}{0.75} \right] = 1.0417.$$

In other words, if D was actually 1,600 units/year, and we made the worst possible mistake in estimating its value (i.e., that D was 900 units/year), then the penalty-cost ratio would be 4.17% of $AC(Q^*)$. Since $AC(Q^* = 400) = \$800/\text{year}$, the management-cost penalty would be $0.0417 \times \$800 = \$33.36/\text{year}$.

Similarly, if we chose $\hat{Q} = 400$ (assuming $D = 1600$) when, in fact, $D = 900$ (and, hence, $Q^* = 300$), then $\hat{Q}/Q^* = 400/300 = 1.333$. The penalty-cost ratio would, again, be 1.0417, as before. However, in this case, $AC(Q^* = 300) = \$600/\text{year}$. Hence, the management-cost penalty would be $0.0417 \times \$600 = \$25.02/\text{year}$.

Based on these calculations, we *have not learned* the what Q^* is, but we *have learned* that if we chose any estimate of D in the range [900, 1,600], then the maximum penalty-cost we would experience would be 4.17% of the minimum possible cost $AC(Q^*)$. Of course, in choosing a value for D to use in a scenario like this, management probably would not choose one of the extreme values (e.g., 900 or 1,600), but, instead, choose an estimate in between. For example, if management's estimated demand rate—we will denote this " \hat{D} " henceforth—was $\hat{D} = 1,200$ units/year, then the maximum penalty-cost ratio would be 1.010. In other words, the corresponding management-cost penalty would be just 1% above the minimum possible (of either \$800 or \$600/year).

Sensitivity Analysis: Step 3

In the Step 3 we extend the assumptions of Step 2, except, now, management is not certain about the exact values for *any* of its inputs. So, in addition to choosing a value for \hat{D} , management also has to choose estimates for K , and h , which we will denote \hat{K} and \hat{h} , respectively.

To take a specific example, suppose management knows only that:

$$\begin{aligned} 900 &\leq D \leq 1,600 \text{ unit per year,} \\ \$75 &\leq K \leq \$125/\text{order, and} \\ \$1.50 &\leq h \leq \$2.50/\text{unit} \times \text{year.} \end{aligned}$$

As in Step 2, in order to compute the maximum management-cost penalty, we have to consider the most extreme estimates ($\hat{D}, \hat{K}, \hat{h}$) that management might choose and the most extreme true values that the (D, K, h) might take on. In order to simplify the process, we can use the fact that the penalty-cost ratio depends only on the ratio \hat{Q}/Q^* ; and, in particular, that, whatever the values of \hat{Q} and Q^* , it is those values that generate either the smallest possible ratio \hat{Q}/Q^* or the largest possible ratio \hat{Q}/Q^* that will yield the largest penalty-cost ratio.

Continuing, we can determine the largest possible \hat{Q}/Q^* by asking: *what is the largest possible \hat{Q} (based on its chosen $\hat{D}, \hat{K}, \hat{h}$,) that management might compute, using the EOQ formula; and what is the smallest that Q^* might possibly be?*

Since EOQ is increasing in D and K and decreasing in h , the largest possible \hat{Q} that management might choose is 516.40 (using $\hat{D} = 1,600$, $\hat{K} = 125$, and $\hat{h} = 1.50$). Similarly, the smallest possible Q^* is 232.39 (based on $D = 900$, $K = 75$, and $h = 2.50$). Hence, the largest possible ratio \hat{Q}/Q^* ratio in this scenario equals 2.22 ($=516.40/232.39$). Using the same logic, the smallest possible ratio \hat{Q}/Q^* ratio in this scenario equals 0.45 ($1/2.22 = 232.39/516.40$).

Plugging either of these \hat{Q}/Q^* ratios into [equation \(7\)](#) yields the largest possible penalty-cost ratio that management might experience, regardless of the value of \hat{Q} it computes (that is, regardless of its estimates of D , K , and h ; and regardless of what the true values of these parameters might be). This maximum ratio is

$$\frac{AC(\hat{Q})}{AC(Q^*)} = 0.5 \times \left[2.22 + \frac{1}{2.22} \right] = 1.335.$$

In managerial terms, the analysis above tells us that in this EOQ business scenario, but with considerable uncertainty about the values of the parameters (D , K , h) to use in the EOQ formula, the *worst* management-cost penalty we would pay would equal 33.5% above the minimum possible cost (i.e., if we *knew* what the precise values of (D , K , h) were).

Finally, as in Step 2, management faced with this scenario would most likely choose in-between values for \hat{D} , \hat{K} , and \hat{h} , not their possible extremes. For example, if management chose to use $\hat{D} = 1200$, $\hat{K} = \$96.82$, and $\hat{h} = \$1.94$, then the maximum management-cost penalty would be 8.11% of $AC(Q^*)$.

The management intuition from Step 3 is that as long as the assumptions of the EOQ business scenario are valid, *one can use the EOQ formula, (4), to compute an order-quantity that results in a relatively small penalty-cost ratio even if management has a great deal of uncertainty about the values of D , K , and h to “plug” into it!*

Sensitivity Analysis: Step 4

Finally, we relax the most significant of the EOQ model’s assumptions: namely, that the demand rate, D , the fixed order cost, K , and the inventory-holding cost, h , are stationary (i.e., never changing).

Suppose, now, that the demand rate D varies over some known range, $[D_{\text{Min}}, D_{\text{Max}}]$. For example, suppose $D_{\text{Min}} = 900$ units/year and $D_{\text{Max}} = 1600$ units/year. Note that the difference between this scenario and the one above is that D is no longer fixed at some unknown value. Instead, D *varies*, perhaps in an unknown manner, within this range. Similarly, suppose that both K and h vary, again, perhaps in some unknown manner, over their respective ranges; i.e., $\$75 \leq K \leq \$125/\text{order}$ and $\$1.5 \leq h \leq \$2.50/\text{unit} \times \text{year}$.

Common sense suggests that since D , K , and h all change their respective values over time, then the optimal order-quantity policy probably should, too. In other words, the optimal policy probably orders different quantities at different

times, depending on how predictable the pattern of changes in D , K , and h are and depending how much management knows about them. Instead, we will use a “stationary” order-quantity policy (i.e., one that orders the same quantity every time), and compute that order-quantity, \hat{Q} , using stationary (i.e., fixed) estimates of their true values.

For the sake of convenience, we will use the values $\hat{D} = 1,200$, $\hat{K} = \$96.82$, and $\hat{h} = \$1.94$, from Step 3. The corresponding $\hat{Q} = 346.09$. Recall from the analysis in Step 3, that regardless of whatever the true stationary values of D , K , and h , this order-quantity will never yield a management-cost penalty larger than 8.11% of $AC(Q^*)$.

To see how this result applies when the values of D , K , and h are changing over time, imagine some moment of time when inventory reaches zero units. As described, management will order $\hat{Q} = 346.09$ units. In order to minimize the rate at which ordering and inventory-holding costs are accumulating *at that moment of time*, management should be ordering the Q^* that corresponds to whatever the true values of D , K , and h are *at that moment of time*. We do not know those values at that moment, but we do know the ranges for each. So, based on the analysis in Step 3, we can conclude that whatever their values are, in ordering \hat{Q} we will never be accumulating a management-cost penalty of more than 8.11% above the minimum possible (associated with whatever Q^* is at that moment of time).

Exactly the same argument applies at *any* moment of time. That is, at any moment of time, inventory-management costs are accumulating at a rate based on management’s chosen (stationary) EOQ policy. If, at the same moment of time, management had been using the optimal policy, inventory-management costs would be accumulating at the minimum possible rate. However, at *no* moment of time will management ever be accumulating an inventory-management-cost penalty at a rate higher than 8.11% of $AC(Q^*)$.

See Lowe and Schwarz (1983) for a formal analysis of this management scenario and the conclusion stated above and interpreted below.

The Bottom Line

The management intuition from the sensitivity analysis above is that in *any* EOQ-like management scenario—one with a demand (or usage) rate, a fixed order cost, and an inventory-holding cost—the EOQ formula will provide order-quantities that have relatively small management-cost penalty percentages, regardless of whether the demand rate is either known or constant and regardless of whether or not its associated ordering and inventory-holding costs are either known or constant. “Relatively small” means that the penalty-cost percentage is small (e.g., 8.11%) compared to the possible ranges of values for D , K , and h .

For Lauren Worth, this means, first, that the EOQ model can be used to make “good” order-quantity decisions—that is, decisions with small management-cost penalty percentages—whether or not Cardinal Hospital’s accounting system is able to yield

accurate estimates of costs and/or demands. Furthermore, the EOQ model will make “good” order-quantity decisions even if the cost and demand inputs to the EOQ model are nonstationary (i.e., change their values over time).

This insight is of tremendous importance: It is what accounts for the use of the EOQ model and EOQ formula in so many real-world inventory-management applications.

Back to the Fundamental Decision

We began this chapter by arguing that the *fundamental* decision about the “order-quantity decision” is *not* “What’s the right quantity?”. Instead, it is: “Under what circumstances does the *choice* of the order-quantity make a lot of difference (in terms of the management-cost penalty)?”. And, we promised that we would address this question using the EOQ model. We are now ready to do so.

Consider managing the inventory of some item, or, practically speaking, a set of items with roughly the same ranges of values for their fixed order cost, inventory-holding cost, and demand rates, respectively. Apply the analysis above in order to determine the maximum inventory-management penalty-cost percentage, assuming that the item’s order-quantity would be chosen using the EOQ formula and “ball-park” estimates of its inputs: \hat{D} , \hat{K} , and \hat{h} . Denote this percentage P%. To illustrate, in Step 4 above, we determined that the maximum management-cost penalty, P%, would be 8.11% of $AC(Q^*)$.

Now, the question is: “Is P% a large dollar amount or not?” For example, is an 8.11% management-cost penalty in Step 4 a significant amount of money or not? The answer, of course, depends on the true, but unknown values of D , K , and h , but we can put an upper limit on it by assuming that the true values of D , K , and h are their maximum possible values: D_{Max} , K_{Max} , and h_{Max} .

In Step 4 these values are $D = 1,600$, $K = \$125$, and $h = \$2.50$. Based on these, the corresponding $AC(Q^*) = \sqrt{2 \times 125 \times 1600 \times 2.50} = \$1,000/\text{year}$. Hence, the maximum possible management-cost penalty incurred by managing this item using the EOQ model is $\$81.10/\text{year} = 0.0811 \times 1,000$.

Finally, management must ask itself: Would the information and implementation costs associated with managing this item using a more sophisticated model—one that took into account the fluctuating values of D , K , and h —cost more or less than $\$81.10/\text{year}$? Remember, the maximum possible reduction in annual ordering and inventory-holding costs resulting from using this more sophisticated model is $\$81.10$.

If the answer is that this more sophisticated model would cost more than $\$81.10/\text{year}$, then management should choose to use the EOQ model to select \hat{Q} . Why? Because even if this more sophisticated model were able to reduce inventory-management costs to the minimum, since it cost more than it was worth, it would actually increase the total management costs associated with this item.

If, on the other hand, the answer is that this more sophisticated model would cost less, then, management should test this model’s performance against that of

the EOQ model. Most likely this will require a computer simulation of the management scenario, one that would keep track of the total inventory-management costs of both models—EOQ and the more sophisticated model—including whatever the increased information and implementation costs of the more sophisticated model are.

In other words, by determining the maximum management-cost penalty/year that is associated with the use of the EOQ order-quantity, management can determine the maximum savings—that is, the maximum possible reduction in penalty cost—that might be derived from the use of a more sophisticated system.

In the same manner, management can estimate the maximum management-cost penalty associated with any order-quantity decision-rule, such as “Order a case.” or “Order a month’s supply.” All that is required are estimates of the maximum and minimum possible values for D , K , and h , as in Steps 3 and 4 above. Then, determine the minimum and maximum values of \hat{Q}/Q^* to determine the penalty-cost ratio $AC(\hat{Q})/AC(Q^*)$. Finally, using the maximum possible value of $AC(Q^*)$, one can determine the corresponding management-cost penalty.

Extensions

The basic EOQ business scenario can be extended in a wide variety of ways. Here are just a few:

Integer and Case Quantities

Equation (4) often results in a Q^* that isn’t an integer. For example, suppose that Lauren Worth computes $Q^* = 346.09$ units for some item she is analyzing. Since supplies can only be ordered in integer quantities, Lauren knows that Capital Hospital’s supplier would laugh at her if she attempted to order 346.09 units. Given the shape of Fig. 8.1, it is obvious that Lauren’s selected \hat{Q} should be either the integer just above or just below the noninteger Q^* , since an even larger or even smaller integer value, respectively, would increase $AC(\hat{Q})$ unnecessarily. For example, given $Q^* = 346.09$, Lauren should order either $\hat{Q} = 346$ or 347 units. But which? Strictly speaking, Lauren *should* evaluate $AC(\hat{Q} = 346)$ and $AC(\hat{Q} = 347)$ and chose the \hat{Q} with the smaller $AC(\hat{Q})$.

However, given the fact that the cost-penalty ratio associated with a \hat{Q} value different from Q^* depends only on how far the ratio of \hat{Q}/Q^* is from 1— see equation (7)— and that, whether we choose $\hat{Q} = 346$ or 347 , the corresponding \hat{Q}/Q^* ratio will be *very close* to 1, Lauren could simply round $Q^* = 346.09$ in the conventional manner (i.e., to $\hat{Q} = 346$) and pay a little or no cost penalty

Another practical consideration in purchasing scenarios is that suppliers may only accept orders for case quantities, where a case contains, say, 12 or 48 units.

Continuing the example above, suppose the EOQ formula prescribes $Q^* = 346.09$ units, but that this item is sold only in cases of 12. Hence, $\hat{Q}^* = 346.09/12 = 28.84$ cases. Again, strictly speaking, Lauren should evaluate $AC(\hat{Q}=28 \text{ cases} = 336 \text{ units})$ and $AC(\hat{Q}=29 \text{ cases} = 348 \text{ units})$ and choose the \hat{Q} with the smaller $AC(\hat{Q})$. Or, equivalently, Lauren should pick the \hat{Q} whose \hat{Q}/Q^* ratio is the closest to 1. However, since, again, both these ratios are very close to 1, Lauren *could* simply round conventionally—that is, round up to 29 cases—and pay a little or no cost penalty.

Other Constraints on Q

In some scenarios there are other constraints on the order quantity. For example, in Cardinal Hospital's business scenario the supplier for some given item might require that some minimum quantity be ordered. To illustrate, suppose in the scenario in which $Q^* = 346.09$, the supplier requires a minimum order of 500 units. In such scenarios, Fig. 8.1 tells Lauren that the best thing to do is to order either Q^* or the supplier's minimum amount, whichever is larger. Hence, in this example, Lauren should select $\hat{Q} = 500$ units (since ordering a larger amount will only yield an $AC(Q)$ larger than $AC(\hat{Q} = 500)$).

In some scenarios, storage considerations may constrain Q to fit into the assigned storage space. In such scenarios, Fig. 8.1 tells us that the best thing to do is to order Q^* or the capacity of the assigned space, whichever is smaller, since ordering an even smaller \hat{Q} will yield higher costs.

Quantity Discounts

In many purchasing scenarios suppliers offer discounts on the cost of the units themselves in order to motivate buyers to purchase in larger quantities. Such "quantity discount" scenarios are straightforward to handle. In particular, the goal remains the same as in the basic EOQ model: to select a value of Q , Q^* , that minimizes $AC(Q)$. However, since the unit cost, denoted c , in the basic EOQ scenario above, now depends on the order-quantity, $AC(Q)$ now must include the cost of a year's supply. In other words, equation (1) and not equation (2) should be used in evaluating $AC(Q)$.

As an illustration, consider the most common quantity discount, the so-called "all-units" discount scheme. Under this scheme if \hat{Q} is below some "breakpoint quantity," denoted "b," then each unit of Q costs a higher price, denoted C_H ; however, if \hat{Q} is greater than or equal to b, then each unit costs a lower per-unit price, denoted C_L .

The simplest way to handle this scenario is to determine the best choice of Q , first, with unit cost C_H , then, with unit cost C_L ; and, finally, select the best overall \hat{Q} . Hence, Lauren would first compute Q_H^* (that is, Q^* if the unit cost is C_H), using

equation (4). Then, based on equation (1), compute $AC(Q_H^*)$. Next, compute Q_L^* (that is, the Q^* if the unit cost is C_L). However, in this step it is important to recognize that in order to qualify for the lower unit cost, C_L , Q_L^* must be greater than or equal to b , the breakpoint quantity. Hence, as described above, the order-quantity Lauren should select is the maximum of Q_L^* and b . Finally, Lauren should order either this quantity or Q_H^* , depending on whose $AC(Q)$ —using equation (1)—is lower.

Finite Planning Horizon

Another major assumption of the EOQ business scenario is that demand goes on forever; in other words that the “planning horizon” is infinite. If, instead, management knows that demand will only last for, say, a fixed T years, then it would be foolish to order EOQ every time inventory fell to zero; and then, if inventory were leftover at the end of T years, to throw away those leftovers.

Schwarz (1972) has shown that, in this business scenario the optimal order-quantity is stationary (i.e., Q^* will never change); that is, management should order an equal quantity n times over T years. Schwarz further demonstrates that the optimal n , denoted n^* satisfies

$$n(n+1) \geq \frac{D \times h \times T^2}{2K} \geq (n-1)n. \tag{8}$$

In order to develop some intuition about n^* , note that, given (8), then n^{*2} is approximately equal to

$$n^{*2} \approx \frac{D \times h \times T^2}{2K} = \frac{D^2 \times T^2}{Q^{*2}},$$

which, in turn, means that n^* is approximately equal to DT/Q^* ; in other words, n^* is approximately equal to the total number of units demanded over the finite horizon, DT , divided by the number of units in an EOQ. Hence, a very good rule-of-thumb for picking n is to determine how many EOQs would satisfy DT , and then round this number to the closest integer.

(Q,r) Policies

In order to gain insight into the order-quantity decision, the EOQ business scenario is structured so that the “order-point decision”—that is, what level of inventory, denoted r , should signal management to order its selected \hat{Q} ?—is a trivial one. In its simplest form, the EOQ business scenario assumes that the order leadtime is zero. However, as long as demand (or usage) over the leadtime is known, the optimal order-point is to order whenever inventory equals a leadtime’s supply.

Unfortunately, there are many real-world scenarios in which management *does not know* what demand during the leadtime will be! This leads to two important questions, one of them of some theoretical significance, and one of them of great practical significance.

The theoretical question is: How should the order-quantity and order-point denoted “ (Q,r) ,” be chosen in order to minimize, average annual cost/time? The practical question is: How “good” are EOQ order-quantity decisions in business scenarios like this; that is, if one selected $Q = EOQ$ and then selected a corresponding r , then would the cost-penalty be large or small?

The theoretical question has been addressed by several researchers, under a variety of different assumptions. For example, consider a scenario where all of the assumptions of the EOQ scenario apply, except that leadtimes are fixed and known, and uncertain customer demand is generated by a probability distribution such as the Poisson. For this scenario, Federgruen and Zheng (1992) describe an algorithm for computing (Q^*, r^*) .

The practical question—which basically asks: How “good” is *EOQ* as the value of Q in a (Q,r) policy?—is an important, since, if the answer was that *EOQ* was “bad” at prescribing the order-quantity, then virtually every implemented real-world (Q,r) policy would be “bad,” too, since *EOQ* is the de facto choice for the order-quantity in virtually every (Q,r) business setting.

The rationale for this de facto choice has been, as we have seen above, that inventory-management cost is insensitive to the choice of the order-quantity decision: insensitive to demand being known or constant, and even insensitive to its fixed ordering and inventory-holding costs being known or constant. Hence, as long as one picked an order-quantity in the “ballpark” of the correct value, one would pay a small cost penalty. The expectation, then, was that something like the same insensitivity applies in selecting the Q in a (Q,r) policy; i.e., that *EOQ* would provide an order-quantity in the ball park.

Zheng (1992) brilliantly validates the use of *EOQ* as a very good choice for Q in one specific (Q,r) scenario. Although the details are beyond the scope of this chapter, we will quote and interpret two of Zheng’s findings: First, “... in contrast to the (basic) *EOQ* model, the controllable costs of the stochastic model due to the selection of the order quantity (assuming the (order point) is chosen optimally for every order quantity) are actually smaller...” What this means is that the cost/time in the (Q,r) model is *less* sensitive to the order-quantity decision than the cost/time in the *EOQ* model. Second, Zheng is able to prove that “...the relative increase of the costs incurred by using the (order-quantity) determined by the *EOQ* instead of the optimal from the stochastic model is no more than 1/8, and vanishes when ordering costs are significant relative to other costs.” What this means is that the cost-penalty ratio associated with selecting Q to be *EOQ* and then selecting the order-point optimally is no more than 1.125; that, at worst, management would incur a 12.5% management-cost penalty as a consequence of its incorrect choice of Q .

Gallego (1998) describes heuristic (Q,r) policies and provides additional references.

Net-Present Value Criterion

The EOQ business scenario uses the criterion of minimizing the average cost/time over an infinite time horizon in determining the optimal order quantity. One alternative to this criterion is net-present value. As applied in the EOQ scenario, we wish to select the order-quantity \hat{Q} that minimizes the net-present value (i.e., discounted total cost) associated with this order quantity.

There are three cost inputs: (1) the fixed order cost K ; and (2) the cost of capital, denoted i , on the dollars invested in the units, and measured in dollars/(dollar \times time); and (3) the cost of the units themselves, denoted c , as above, and measured in dollars/unit.

It can be shown, see Zipkin (2000) for example, that the optimal Q in this scenario, denoted “ Q_N^* ,” is approximately equal to EOQ; that is,

$$Q_N^* \approx \sqrt{\frac{2K \times D}{i \times c}}.$$

Application: Cash Management

The EOQ model can be applied to *any* quantity decision that is repeated over time, where there is a trade-off between a fixed cost (i.e., EOQ’s order cost K) and a variable cost associated with that decision (i.e., EOQ’s inventory-holding cost), and where the choice is based on minimizing the cost/time associated with that decision. The most popular such application involves the management of cash.

The simplest version of a cash-management scenario corresponds to the basic EOQ scenario, but with its inputs reinterpreted. In particular: (1) D is the rate at which cash is being accumulated from some business activity into a noninterest bearing account (e.g., checking account); (2) h is the interest being paid in an interest-bearing account (e.g., money-market account), and is measured in dollars/(dollar \times time); and (3) K is the fixed cost incurred in transferring any amount of money, Q , from the non-interest to the interest-bearing account. Hence, Q^* from (4) is the optimal transfer quantity.

Historical Background

The EOQ model and formula are attributed to Ford Whitman Harris (Harris, 1913), and originally published in 1913 in *Factory, The Magazine of Management*. However, despite this magazine’s wide circulation among manufacturing managers, Harris’ article had been lost until it was found by Erlenkotter (See Erlenkotter 1989).

In the years between Harris' article being lost and found, the EOQ model, and, in particular, the square-root equation, (4), was attributed to others, some calling it the "Wilson lot size formula," others calling it "Camp's formula." Erlenkotter (1989) describes the early development of the EOQ literature and sketches the life of Harris as an engineer, inventor, author, and patent attorney.

Building Intuition The EOQ formula provides near-optimal order quantities—that is, order quantities that have small management-cost penalties—even in realistic management scenarios when the unrealistic assumptions of the EOQ model don't apply. In particular, the EOQ formula will provide near-optimal order quantities regardless of whether the item's demand rate is either known or constant and regardless of whether or not its associated ordering and inventory-holding costs are either known or constant.

More broadly, the notion of a management-cost penalty introduced in this chapter is an important aid in thinking strategically about the adoption of *any* management system. That is, if we take it as granted that better information and more sophisticated decision-making systems yield better performing (i.e., lower cost) operations, then question is not: *When* or *how* to acquire/adapt them? Instead, the question is: *Are these more sophisticated systems worth it?*

References

- Erlenkotter, D. (1989) "An Early Classic Misplaced: Ford W. Harris's Economic Order Quantity Model of 1915," *Management Science* 35:7, pp. 898–900.
- Federgruen, A. and Y. S. Zheng. (1992) "An Efficient Algorithm for Computing an Optimal (r,Q) Policy in Continuous Review Stochastic Inventory Systems," *Operations Research* 40:4, pp. 808–813.
- Gallego, G. (1998) "New Bounds and Heuristics for (Q,r) Policies," *Management Science* 44:2, 219–233.
- Harris, F. M. (1913) "How Many Parts to Make at Once," *Factory, The Magazine of Management* 10:2, 135–136, 152. Reprinted in *Operations Research* 38:6 (1990), 947–950.
- Lowe, T. J. and L. B. Schwarz. (1983) "Parameter Estimation for the EOQ Lot-Size Model: Minimax and Expected-Value Choices," *Naval Research Logistics Quarterly*, 30, 367–376.
- Schwarz, L. B. (1972) "Economic Order Quantities for Products with Finite Demand Horizons," *AIIE Transactions* 4:3, 234–237.
- Zheng, Y. S. (1992) "On Properties of Stochastic Inventory Systems," *Management Science* 38:1, 87–103.
- Zipkin, P. (2000) *Foundations of Inventory Management*, McGraw-Hill Higher Education, New York, New York.